Student Number:

Set:

Shore

Year 12
Mathematics Extension 1
Trial Examination
2008

General Instructions

- Reading time − 5 minutes
- Working time − 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

Note: Any time you have remaining should be spent revising your answers.

Total marks − 84

- Attempt Questions 1 − 7
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Total Marks − 84
Attempt Questions 1−7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) **Marks**

(a) Find the size of the acute angle between the two lines $y = 2x - 3$ and $y = 4 - x$. Leave your answer correct to the nearest degree. **2**

(b) Find a primitive of $\int \frac{5}{\sqrt{4 - x^2}}\, dx$. **2**

(c) A family of two parents and four children is being arranged in a line for a family photograph. **2**

Calculate the number of possible arrangements if the two parents are placed at opposite ends of the line.

(d) Solve for x: $\frac{x}{8 - x} \le 1$. **3**

(e) Consider the polynomial given by $P(x) = x^3 - 12x + 16$.

(i) Show that $(x - 2)$ is a factor of $P(x)$. **1**

(ii) Solve $P(x) = 0$. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

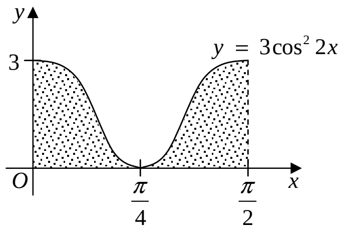
(a) Sketch the graph of $y = 2\cos^{-1} x$, clearly indicating the domain and range of the function. **2**

(b) How many different arrangements are there of the letters in the word NARRABRI? **2**

(c) Solve $\sin 2x - \sin x = 0$ for $0 \le x \le 2\pi$. **2**

(d) Evaluate $\int_0^1 x\left(x^2+1\right)^5 dx$ using the substitution $u = x^2 + 1$. **3**

(e) **3**



Not to Scale

The diagram above shows the graph of $y = 3\cos^2 2x$ for $0 \le x \le \frac{\pi}{2}$.

Find the value of the shaded area bounded by the curve $y = 3\cos^2 2x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) A project team consisting of 2 engineers and 3 draftspersons is to be formed for a particular project. There are 6 engineers and 9 draftspersons available for selection. How many different project teams can be formed? **2**

(b) Use the expansion of $\cos(A + B)$ to show that **2**

$$\cos 105^\circ = \frac{\sqrt{2} - \sqrt{6}}{4}.$$

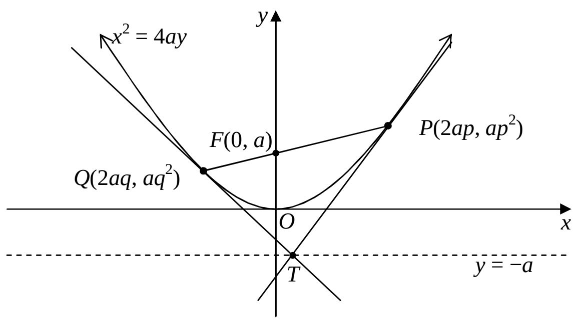
(c) A term of the expansion of $(2 + mx)^{12}$ is $24057x^5$. Find the value of m. **2**

(d) Use the principle of mathematical induction to prove that **3**

$$4 + 16 + 64 + \ldots + 4^n = \frac{4^{n+1} - 4}{3}$$

for all integers $n \ge 1$.

(e)



Not to Scale

The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ such that PQ is a focal chord with equation $y = \left(\frac{p+q}{2}\right)x - apq$. The focus, F, has coordinates $(0, a)$ and O is the origin.

(i) Show that $pq = -1$. **1**

(ii) The tangents to the parabola at P and Q have equations $y = px - ap^2$ and $y = qx - aq^2$ respectively. The tangents intersect at the point T. Show that T lies on the directrix $y = -a$. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) A particle is moving such that its displacement x metres from a fixed point O at any time t seconds is given by

$$x = 2\cos 3t - \sqrt{12}\sin 3t .$$

(i) Prove that the motion is simple harmonic by showing that $\ddot{x} = -9x$. **2**

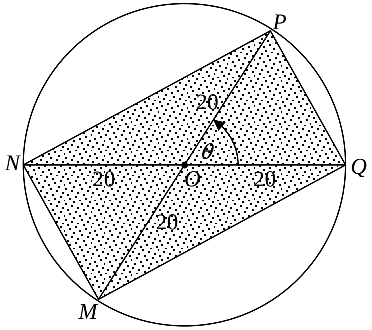
(ii) Find exact values for R and α such that **2**

$$2\cos 3t - \sqrt{12}\sin 3t \equiv R\cos(3t+\alpha)$$

where $R > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$.

(iii) Hence, find the first time when the particle is at the centre of motion. **2**

(b)



Not to Scale

A rectangle $NPQM$ is inscribed in a circle of radius 20 centimetres with centre at O such that NQ and MP are diameters. The angle, θ radians, between these two diameters is increasing at the rate of 0.1 radians per second.

(i) Show that the area of the rectangle, A square centimetres, is given by **2**

$$A = 800\sin\theta .$$

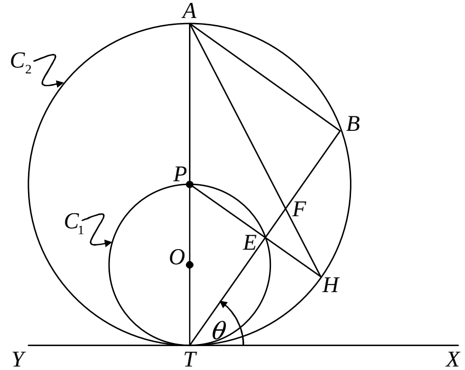
(ii) Calculate the rate at which the area A is changing when the angle between the diameters is $\dfrac{\pi}{6}$ radians. **2**

(c) By applying the binomial theorem to $(1+x)^{2n}$ and differentiating, show that **2**

$$n \times 4^n = {}^{2n}C_1 + 2\,{}^{2n}C_2 + \ldots + k\,{}^{2n}C_k + \ldots + 2n\,{}^{2n}C_{2n} .$$

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) **3**



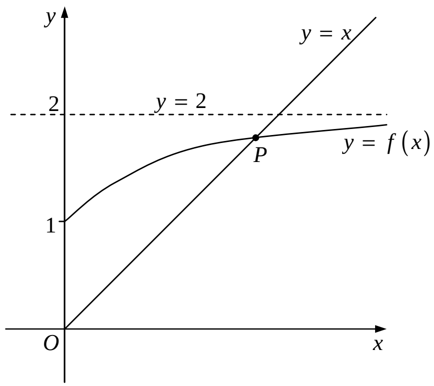
Not to Scale

In the diagram, O is the centre of the smaller circle C_1. P is a point on the circle C_1 and the centre of the larger circle C_2. XY is a common tangent to both the circles at the point T. The diameter PT of circle C_1 is produced to meet circle C_2 at the point A. The points B and H lie on the circle C_2. BT meets circle C_1 at the point E and AH at the point F. Let $\angle BTX = \theta$.

Show that AH bisects $\angle TAB$.

Question 5 continues

(b)



Not to Scale

A function is defined as $f(x) = 2 - e^{-x}$ for $x \geq 0$. The diagram shows the graph of $y = f(x)$, the lines $y = x$ and $y = 2$. The curve $y = f(x)$ and the line $y = x$ intersect at the point P.

Copy or trace the diagram onto your writing booklet.

(i) State the domain of the inverse function $y = f^{-1}(x)$. **1**

(ii) On your diagram, sketch the inverse function $y = f^{-1}(x)$. **2**

(iii) Show that the x-coordinate of the point P is a solution of the equation **1**

$$x + e^{-x} - 2 = 0.$$

(iv) A first approximation to the solution of the equation $x + e^{-x} - 2 = 0$ is $x = 1.8$. Use one application of Newton's method to find a better approximation of the x-coordinate of P. Give your answer correct to four decimal places. **2**

(v) Hence, approximate the area enclosed between the function $y = f(x)$, its inverse $y = f^{-1}(x)$ and the coordinate axes. Leave your answer correct to 4 decimal places. **3**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

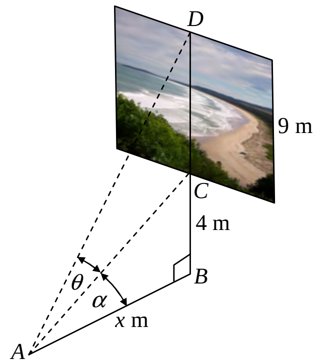
(a) A particle moves so that its acceleration at any time t seconds is given by $\ddot{x} = -8e^{-4x}$. Initially the particle is at the origin, O, with velocity 2 m/s.

(i) Show that the velocity of the particle at any time t is given by $\dot{x} = 2e^{-2x}$. **3**

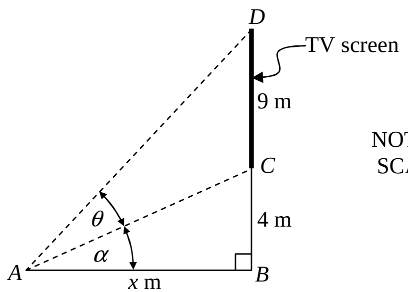
(ii) Hence, show that the displacement of the particle at any time t is given by $x = \frac{1}{2}\ln(4t + 1)$. **3**

(b) A rectangular TV screen is set up in a level field for spectators to watch a movie. The screen is mounted 4 metres off the ground and is 9 metres high and 16 metres wide.

Tom sits directly in front of the centre of the screen at the point A which is x metres from the point B directly below the screen as shown in the diagrams below. The angle DAC through which he views the bottom of the screen C and the top of the screen D is θ radians as shown.



3D view



Side view

NOT TO SCALE

Let angle CAB be α radians.

(i) By considering the expansion of $\tan(\theta + \alpha)$, show that **3**

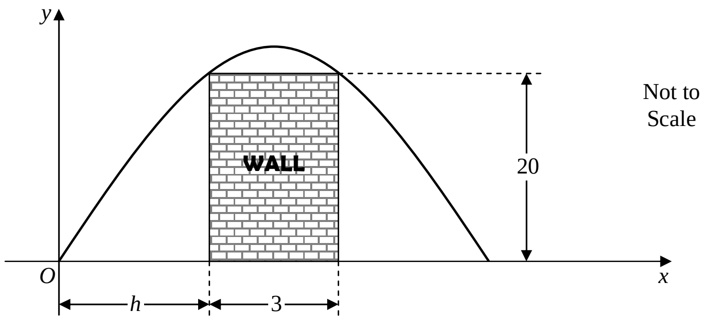
$$\theta = \tan^{-1}\left(\frac{9x}{52 + x^2}\right).$$

(ii) Hence, or otherwise, find the value of x (correct to 2 decimal places) for which Tom has the maximum viewing angle θ. Justify your answer. **3**

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) A six-sided die is weighted so that the probability of rolling a "six" is twice that of any other number. Find the probability of rolling at least nine "sixes" in 10 rolls of this die. [You do not need to simplify your answer.] **3**

(b) The wall of a fort on level ground is 3 metres thick and 20 metres high. A projectile is fired from a point O outside the fort, h metres from the base of the wall of the fort, towards the fort as shown in the diagram below.



It is assumed that the path of the projectile traces out a parabola of the form $y = bx - ax^2$ where a and b are constants.

(i) Show that $b = \dfrac{20(2h+3)}{h(h+3)}$ and $a = \dfrac{20}{h(h+3)}$. **3**

$\left[\text{i.e. Show that } y = \dfrac{20(2h+3)x}{h(h+3)} - \dfrac{20x^2}{h(h+3)}.\right]$

(ii) Let the angle of projection of the projectile be θ degrees and the initial velocity be V m/s and the constant of gravity be $g = 10$. **1**

Hence the equations of motion are $x = Vt\cos\theta$ and $y = Vt\sin\theta - 5t^2$.

Show that the equation of the path of flight of the projectile is given by

$$y = x\tan\theta - \frac{5x^2}{V^2\cos^2\theta}.$$

(iii) Hence, show that $V^2\cos^2\theta = \dfrac{h(h+3)}{4}$. **1**

(iv) If the projectile is fired at an angle of 45°, find the values of h and V correct to 2 decimal places. **4**

End of Paper

YEAR 12 EXT 1 MATHS – Trial Exam 2008

1 a) $m_1 = 2$, $m_2 = -1$

$\tan\theta = \left|\dfrac{2 - -1}{1 + 2\times -1}\right|$

$\tan\theta = \left|\dfrac{3}{-1}\right|$

$\theta = \tan^{-1} 3$

Angle $= 71°33'54.18''$

$\doteqdot 72°$ (nearest degree)

b) $\displaystyle\int \frac{5}{\sqrt{4-x^2}}\,dx = 5\int \frac{1}{\sqrt{4-x^2}}\,dx$

$= 5\sin^{-1}\left(\frac{x}{2}\right) + C$

c) $\boxed{2}\times\boxed{4}\times\boxed{3}\times\boxed{2}\times\boxed{1}\times\boxed{1}$

P C C C C P

Arrangements $= 2!\times 4!$

$= 48$

d) $\dfrac{x}{8-x} \leq 1$

critical points

1) $\boxed{x = 8}$

2) $\dfrac{x}{8-x} = 1$

$x = 8 - x$

$2x = 8$

$\boxed{x = 4}$

$\therefore x \leq 4$ or $x > 8$

1 e) i) $P(2) = 2^3 - 12\times 2 + 16$

$= 0$

$\therefore (x-2)$ is a factor.

ii) $P(x) = (x-2)(x^2+2x-8)$

$= (x-2)(x+4)(x-2)$

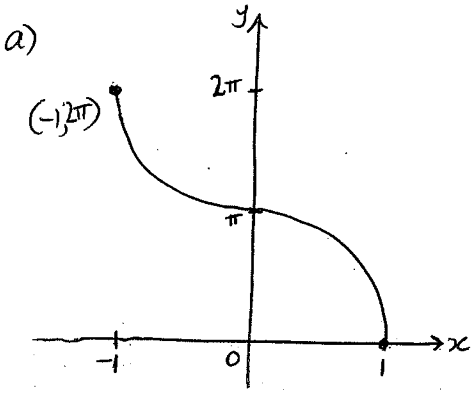
let $P(x) = 0$

$0 = (x-2)^2(x+4)$

$\therefore x = 2$ or $x = -4$.

Question 2

a)



b) arrangements $= \dfrac{8!}{3!\,2!}$

$= 3360$

c) $\sin 2x - \sin x = 0$

$2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$ or $\cos x = \frac{1}{2}$

$x = 0, \pi, 2\pi$ or $\frac{\pi}{3}, 2\pi - \frac{\pi}{3}$

$x = 0, \pi, 2\pi, \frac{\pi}{3}$ or $\frac{5\pi}{3}$

2 d) $\int_0^1 x(x^2+1)^5 dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$$\boxed{\begin{array}{ll} \text{limits} & \\ x=0 & x=1 \\ u=1 & u=2 \end{array}}$$

$\therefore \frac{1}{2}\int_0^1 2x(x^2+1)^5 dx = \frac{1}{2}\int_1^2 u^5 du$

$= \frac{1}{2}\left[\frac{u^6}{6}\right]_1^2$

$= \frac{1}{2}\left(\frac{2^6}{6} - \frac{1^6}{6}\right)$

$= 5\frac{1}{4}$

e). Curve is symmetrical about $x = \frac{\pi}{4}$.

$\therefore$ Area $= 2\int_0^{\frac{\pi}{4}} 3\cos^2 2x\, dx$

$= 6\int_0^{\frac{\pi}{4}} \cos^2 2x\, dx$

$$\boxed{\begin{array}{l} \text{Consider} \\ \cos 2A = 2\cos^2 A - 1 \\ \therefore \cos^2 A = \dfrac{\cos 2A + 1}{2} \\ \therefore \cos^2 2x = \dfrac{\cos 4x + 1}{2} \end{array}}$$

2e) cont'd....

Area $= 6\int_0^{\frac{\pi}{4}} \frac{\cos 4x + 1}{2} dx$

$= 3\int_0^{\frac{\pi}{4}} (\cos 4x + 1)\, dx$

$= 3\left[\frac{1}{4}\sin 4x + x\right]_0^{\frac{\pi}{4}}$

$= 3\left(\left(\frac{1}{4}\sin\frac{4\pi}{4} + \frac{\pi}{4}\right) - \left(\frac{1}{4}\sin 0 + 0\right)\right)$

$= \frac{3\pi}{4}$ units^2

QUESTION 3

a). teams $= {}^6C_2 \times {}^9C_3$

$= 1260$

b). $\cos(A+B) = \cos A\cos B - \sin A \sin B$

$\therefore \cos 105° = \cos(60° + 45°)$

$= \cos 60° \cos 45° - \sin 60° \sin 45°$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$

3 c) $(2+mx)^{12}$

$$24057x^5 = {}^{12}C_{\boxed{5}}\, 2^{\triangle}(mx)^{\boxed{5}}$$

$$= 792 \times 2^7 m^5 x^5$$

$$\therefore 24057 = 101376m^5$$

$$m^5 = \frac{243}{1024}$$

$$\therefore m = \frac{3}{4}$$

d) (*) Prove true for $n=1$

$$LHS = 4^1$$
$$= 4$$
$$RHS = \frac{4^{1+1}-4}{3}$$
$$= \frac{12}{3}$$
$$= 4$$
$$= LHS$$

$\therefore$ true for $n=1$

(*) Assume true for $n=k$

i.e. $4+16+64+\dots+4^k = \dfrac{4^{k+1}-4}{3}$

(*) Prove true for $n=k+1$

i.e. prove $4+16+64+\dots+4^k+4^{k+1} = \dfrac{4^{k+2}-4}{3}$

$$LHS = \underbrace{4+16+64+\dots+4^k}+4^{k+1}$$

$$= \frac{4^{k+1}-4}{3} + 4^{k+1} \quad \text{by assumption}$$

$$= \frac{4^{k+1}-4+3.4^{k+1}}{3}$$

$$= \frac{4.4^{k+1}-4}{3}$$

3 d) cont'd...

$$LHS = \frac{4^{k+2}-4}{3}$$
$$= RHS$$

Hence, if it is true for $n=k$, then it is true for $n=k+1$. Since it is true for $n=1$, then it is true for $n=1+1=2$, and also for $n=2+1=3$, and so on for all integers $n \geq 1$.

3 e) i) Since $F(0,a)$ lies on focal chord, subst. $(0,a)$ into eqn of chord.

$$a = \left(\frac{p+q}{2}\right) \times 0 - apq$$

$$\frac{a}{-a} = \frac{-apq}{-a}$$

$$pq = -1$$

ii) Solve tangents simultaneously

$$px - ap^2 = qx - aq^2$$
$$px - qx = ap^2 - aq^2$$
$$(p-q)x = a(p-q)(p+q)$$
$$x = a(p+q)$$

sub into $y = px - ap^2$

$$y = p(a(p+q)) - ap^2$$
$$= ap^2 + apq - ap^2$$
$$= apq$$

but $pq = -1$ (part (i))

$\therefore y = -a$ which means T lies on directrix.

QUESTION 4

a) i) $x = 2\cos 3t - \sqrt{12}\sin 3t$

$\dot{x} = -6\sin 3t - 3\sqrt{12}\cos 3t$

$\ddot{x} = -18\cos 3t + 9\sqrt{12}\sin 3t$

$= -9(2\cos 3t - \sqrt{12}\sin 3t)$

$\therefore \ddot{x} = -9x$ as $x = 2\cos 3t - \sqrt{12}\sin 3t)$

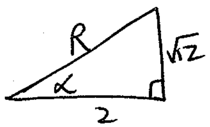
ii) let $2\cos 3t - \sqrt{12}\sin 3t \equiv R\cos(3t+\alpha)$

$= R\cos 3t\cos\alpha - R\sin 3t\sin\alpha$

equating coefficients

$\underbrace{R\cos\alpha = 2 \quad \cos\alpha = \frac{2}{R} \qquad\qquad R\sin\alpha = \sqrt{12} \quad \sin\alpha = \frac{\sqrt{12}}{R}}$

$\therefore$



By pythag

$R^2 = \sqrt{12}^2 + 2^2$

$\therefore R = 4$

$\tan\alpha = \frac{\sqrt{12}}{2}$

$\alpha = \frac{\pi}{3}$

$\therefore 2\cos 3t - \sqrt{12}\sin 3t \equiv 4\cos(3t + \frac{\pi}{3})$

iii) At centre of motion when $x = 0$

$4\cos(3t + \frac{\pi}{3}) = 0$

$\cos(3t + \frac{\pi}{3}) = 0$

$\therefore 3t + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$3t = \frac{\pi}{6}, \frac{7\pi}{6}, \ldots$

$t = \frac{\pi}{18}, \ldots$

$\therefore$ First at centre of motion after $\frac{\pi}{18}$ seconds.

4(b) i). Area of $\Delta POQ = \frac{1}{2} \times 20 \times 20 \sin\theta$

$= 200\sin\theta$

Area of $\Delta NOP = \frac{1}{2} \times 20 \times 20 \sin(180-\theta)$

$= \frac{1}{2} \times 20 \times 20 \sin\theta$ (since $\sin\theta = \sin(180-\theta)$)

$= 200\sin\theta$

$\therefore$ Area Rectangle $= 2 \times$ Area $\Delta POQ + 2 \times$ Area ΔNOP

$= 2 \times 200\sin\theta + 2 \times 200\sin\theta$

$\therefore A = 800\sin\theta$.

ii) $\frac{d\theta}{dt} = 0.1$ (given)

need $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$\frac{dA}{dt} = 800\cos\theta \times 0.1$

$= 80\cos\theta$

When $\theta = \frac{\pi}{6}$

$\frac{dA}{dt} = 80\cos\frac{\pi}{6}$

$= \frac{80\sqrt{3}}{2}$

$\therefore$ Area increasing at $40\sqrt{3}$ cm^2/second

(4) c) $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + {}^{2n}C_3 x^3 + \dots + {}^{2n}C_k x^k + \dots + {}^{2n}C_{2n} x^{2n}$

Differentiate both sides w.r.t. x.

$2n(1+x)^{2n-1} = 0 + {}^{2n}C_1 + 2\,{}^{2n}C_2 x + 3\,{}^{2n}C_3 x^2 + \dots + k\,{}^{2n}C_k x^{k-1} + \dots + 2n\,{}^{2n}C_{2n} x^{2n-1}$

subst $x = 1$

$2n(1+1)^{2n-1} = {}^{2n}C_1 + 2\,{}^{2n}C_2(1) + 3\,{}^{2n}C_3(1)^2 + \dots + k\,{}^{2n}C_k(1)^{k-1} + \dots + 2n\,{}^{2n}C_{2n}(1)^{2n-1}$

$2n \times 2^{2n-1} = {}^{2n}C_1 + 2\,{}^{2n}C_2 + 3\,{}^{2n}C_3 + \dots + k\,{}^{2n}C_k + \dots + 2n\,{}^{2n}C_{2n}$

$n \times 2^1 \times 2^{2n-1} = {}^{2n}C_1 + 2\,{}^{2n}C_2 + 3\,{}^{2n}C_3 + \dots \qquad \dots + 2n\,{}^{2n}C_{2n}$

$n \times 2^{2n} = \dots\dots$

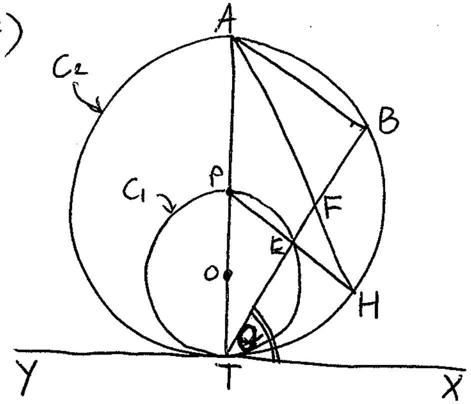
$n \times 4^n = {}^{2n}C_1 + 2\,{}^{2n}C_2 + 3\,{}^{2n}C_3 + \dots \qquad \dots + 2n\,{}^{2n}C_{2n}$

QUESTION 5

see over...

QUESTION 5

a)



$\angle TAB = \angle BTX$ (In circle C_2, angle between tangent and chord drawn to point of contact is equal to angle in alternate segment)
$= \theta$

Likewise in circle C_1,

$\angle TPH = \angle BTX$
$= \theta$.

Now,

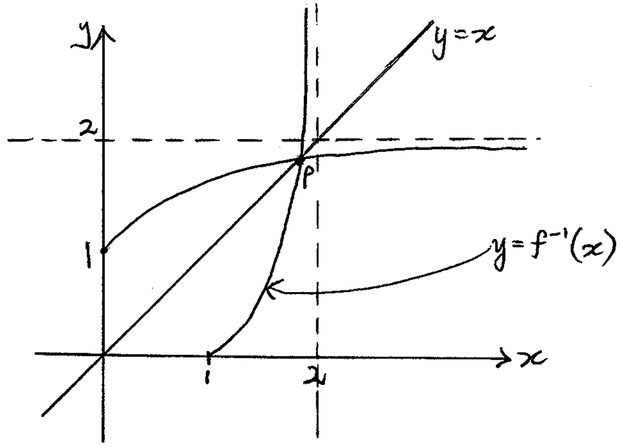
$\angle TPH = 2 \times \angle TAH$ (angle at centre of circle C_2 is equal to twice angle at circumference standing on same arc)

$\therefore \angle TAH = \dfrac{\theta}{2}$

$\therefore \angle TAH = \frac{1}{2} \times \angle TAB$

$\therefore$ AH bisects $\angle TAB$.

5 b)



i) Domain of $y = f^{-1}(x)$ is $1 \leqslant x < 2$.

ii) P lies on point of intersection of $y = x$ and $y = f(x)$

Solving simultaneously,

$$x = 2 - e^{-x}$$

$$\therefore x + e^{-x} - 2 = 0$$

iii) let $g(x) = x + e^{-x} - 2$

$$\therefore g'(x) = 1 - e^{-x}$$

Better approx of x-coord of P

$$x_1 = 1 \cdot 8 - \frac{1 \cdot 8 + e^{-1 \cdot 8} - 2}{1 - e^{-1 \cdot 8}}$$

$$= 1 \cdot 84157 \ldots$$

$$\doteqdot 1 \cdot 8416 \quad (4 \text{dp})$$

5 b) cont'd...

v) $$\text{Area} = 2 \times \int_0^{1 \cdot 8416} \left(2 - e^{-x} - x\right) dx$$

$$= 2 \left[2x + e^{-x} - \frac{x^2}{2}\right]_0^{1 \cdot 8416}$$

$$= 2 \left[\left(2 \times 1 \cdot 8416 + e^{-1 \cdot 8416} - \frac{1 \cdot 8416^2}{2}\right) - \left(0 + e^0 - 0\right)\right]$$

$$= 2 \left[2 \cdot 146 \ldots - 1\right]$$

$$= 2 \cdot 29203 \ldots$$

$$\doteqdot 2 \cdot 2920 \text{ units}^2 \quad (4 \text{dp})$$

QUESTION 6

a) i) $$\frac{d}{dx}\left(\frac{1}{2} v^2\right) = \ddot{x}$$

$$\frac{d}{dx}\left(\frac{1}{2} v^2\right) = -8e^{-4x}$$

$$\frac{1}{2} v^2 = \int -8e^{-4x} \, dx$$

$$\frac{1}{2} v^2 = 2e^{-4x} + C$$

When $x = 0$, $v = 2$

$$\frac{1}{2} \times 2^2 = 2e^0 + C$$

$$2 = 2 + C$$

$$0 = C$$

⑥ a) i) cont'd...

$$\therefore \quad \frac{1}{2}v^2 = 2e^{-4x}$$

$$v^2 = 4e^{-4x}$$

$$v = \pm\sqrt{4e^{-4x}}$$

But when $x=0$, $v>0$

$$\therefore \quad v = \sqrt{4e^{-4x}}$$

$$= 2(e^{-4x})^{\frac{1}{2}}$$

$$\therefore \quad v = 2e^{-2x}$$

⑥ a) ii)

$$\frac{dx}{dt} = 2e^{-2x}$$

$$\frac{dx}{dt} = \frac{2}{e^{2x}}$$

$$\frac{dt}{dx} = \frac{e^{2x}}{2}$$

$$t = \int \frac{e^{2x}}{2}\,dx$$

$$t = \frac{1}{2}\times\frac{1}{2}e^{2x} + c_1$$

when $t=0$, $x=0$

$$\therefore \quad 0 = \frac{1}{4}e^0 + c_1$$

$$\therefore \quad c_1 = -\frac{1}{4}$$

Hence $t = \frac{1}{4}e^{2x} - \frac{1}{4}$

$$4t = e^{2x} - 1$$

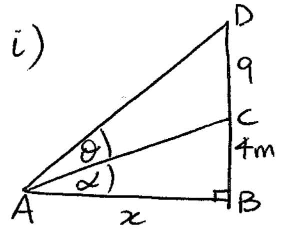
$$4t + 1 = e^{2x}$$

⑥ a) ii) cont'd...

$$\log_e(4t+1) = 2x$$

$$\therefore \quad x = \frac{1}{2}\ln(4t+1)$$

⑥ b) i)



In ΔCAB, $\tan\alpha = \frac{4}{x}$

In ΔBAD, $\tan(\theta+\alpha) = \frac{13}{x}$

Consider $\tan(\theta+\alpha) = \dfrac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$

$$\therefore \quad \frac{13}{x} = \frac{\tan\theta + \frac{4}{x}}{1 - \frac{4\tan\theta}{x}} \times \frac{x}{x}$$

$$\frac{13}{x} = \frac{x\tan\theta + 4}{x - 4\tan\theta}$$

$$13(x - 4\tan\theta) = x(x\tan\theta + 4)$$

$$13x - 52\tan\theta = x^2\tan\theta + 4x$$

$$9x = x^2\tan\theta + 52\tan\theta$$

$$9x = \tan\theta\,(x^2+52)$$

$$\frac{9x}{x^2+52} = \tan\theta$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{9x}{x^2+52}\right)$$

6 b) ii)

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{9x}{x^2+52}\right)^2} \times \frac{(x^2+52).9 - 9x(2x)}{(x^2+52)^2}$$

Maximum value of θ when $\frac{d\theta}{dx} = 0$

$$0 = \frac{9(x^2+52) - 18x^2}{\left(1+\left(\frac{9x}{x^2+52}\right)^2\right)(x^2+52)^2}$$

$$0 = 9x^2 + 468 - 18x^2$$

$$0 = 468 - 9x^2$$

$$= 9(52 - x^2)$$

$$\therefore x^2 = 52$$

$$x = \pm\sqrt{52}$$

But x is a length and must be positive.

test $x = \sqrt{52}$

		7.21	
x	7	$\sqrt{52}$	8
$\frac{d\theta}{dx}$	0.101...	0	-0.004

$\therefore$ maximum viewing angle when $x = \sqrt{52}$

ie. $x \doteqdot 7.21$ (2dp)

QUESTION 7

7 a) $P\left(\begin{smallmatrix}\text{one roll,}\\ \text{one six}\end{smallmatrix}\right) = \frac{2}{7}$

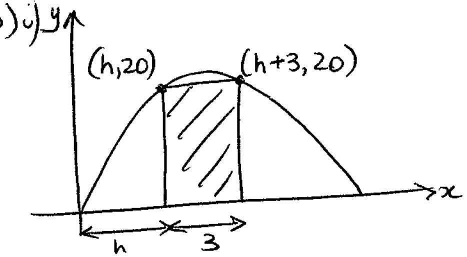
$$P\left(\begin{smallmatrix}\text{at least 9}\\ \text{sixes}\end{smallmatrix}\right) = {}^{10}C_{10}\left(\tfrac{2}{7}\right)^{10} + {}^{10}C_9\left(\tfrac{2}{7}\right)^9\left(\tfrac{5}{7}\right)^1$$

Alternative interpretation:

$P\left(\begin{smallmatrix}\text{one roll,}\\ \text{one six}\end{smallmatrix}\right) = \frac{2}{3}$

$$P\left(\begin{smallmatrix}\text{at least}\\ \text{9 sixes}\end{smallmatrix}\right) = {}^{10}C_{10}\left(\tfrac{2}{3}\right)^{10} + {}^{10}C_9\left(\tfrac{2}{3}\right)^9\left(\tfrac{1}{3}\right)^1$$

7 b) i)



parabola: $y = bx - ax^2$

subst point $(h,20)$

$$20 = bh - ah^2$$

$$\therefore b = \frac{20+ah^2}{h} \quad \text{— (1)}$$

subst point $(h+3, 20)$

$$20 = b(h+3) - a(h+3)^2$$

$$20 = bh + 3b - ah^2 - 6ah - 9a \quad \text{— (2)}$$

subst eqn (1) into eqn (2)

$$20 = h\left(\frac{20+ah^2}{h}\right) + 3\left(\frac{20+ah^2}{h}\right) - ah^2 - 6ah - 9a$$

$$\cancel{20} = \cancel{20} + \cancel{ah^2} + \frac{60}{h} + 3ah - \cancel{ah^2} - 6ah - 9a$$

$$0 = \frac{60}{h} - 3ah - 9a$$

$$3ah + 9a = \frac{60}{h}$$

$$a(3h+9) = \frac{60}{h}$$

7 b) i) cont'd...

$$a = \frac{60}{h(3h+9)}$$

$$a = \frac{60}{3h(h+3)}$$

$$a = \frac{20}{h(h+3)}$$

subst into eqn 1

$$b = \frac{20 + \left(\frac{20}{h(h+3)}\right)h^2}{h}$$

$$hb = 20 + \frac{20h}{h+3}$$

$$hb = \frac{20(h+3)+20h}{h+3}$$

$$b = \frac{20h+60+20h}{h(h+3)}$$

$$b = \frac{40h+60}{h(h+3)}$$

$$= \frac{20(2h+3)}{h(h+3)}$$

7 b) ii) $x = Vt\cos\theta$

$$\therefore t = \frac{x}{V\cos\theta}$$

subst into $y = Vt\sin\theta - 5t^2$

$$y = V \cdot \frac{x}{V\cos\theta} \cdot \sin\theta - 5\left(\frac{x}{V\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{5x^2}{V^2\cos^2\theta}$$

b) iii) Equating coefficients of x^2 in parts (i) & (ii)

$$\frac{20}{h(h+3)} = \frac{5}{V^2\cos^2\theta}$$

$$20V^2\cos^2\theta = 5h(h+3)$$

$$V^2\cos^2\theta = \frac{h(h+3)}{4}$$

iv) If $\theta = 45°$

then $V^2\cos^2 45° = \frac{h(h+3)}{4}$

$$V^2\left(\frac{1}{2}\right) = \frac{h(h+3)}{4}$$

$$V^2 = \frac{h(h+3)}{2} \quad \text{——— } A$$

7 b) iv) cont'd...

Equating coeffs of x in parts (ii) & (i)

$$\tan\theta = \frac{20(2h+3)}{h(h+3)}$$

But $\theta = 45^\circ$

$$\tan 45^\circ = \frac{20(2h+3)}{h(h+3)}$$

$$1 = \frac{20(2h+3)}{h(h+3)}$$

$$h^2 + 3h = 40h + 60$$

$$h^2 - 37h - 60 = 0$$

$$h = \frac{37 \pm \sqrt{37^2 - 4\times1\times-60}}{2}$$

$$= \frac{37 \pm \sqrt{1609}}{2}$$

But h is a length and must be positive.

$$\therefore h = \frac{37+\sqrt{1609}}{2}$$

$$= 38.556\ldots$$

Subst this into eqn (A)

$$v^2 = \frac{38.556\ldots(38.556\ldots+3)}{2}$$

$$v^2 = 801.123\ldots$$

$$\therefore v = \pm\sqrt{801.123\ldots}$$

$\therefore v = 28.304\ldots$ (as velocity >0 initially)